Information About Ellipses

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1 Discussion

This document contains various facts about ellipses in the xy-plane. The terminology used is from the book *Calculus and Analytic Geometry, 7th edition* by George B. Thomas, Jr. and Ross L. Finney, Addison-Wesley Publishing Company, Reading, Massachusetts, 1988.

Geometric Definition. An *ellipse* is the set of points in a plane whose distances from two fixed points in that plane add to a constant. One of the fixed points is called a *focal point* of the ellipse. The two together are referred to as the *foci* of the ellipse.

Standard Form. Let the foci be $(\pm c, 0)$ where $c > 0$. Let (x, y) be an ellipse point and let the sum of the distances from (x, y) to the foci be denoted $2a$ for $a > 0$. The equation that (x, y) must satisfy is

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a.$$

The points (x, y), $(c, 0)$, and $(-c, 0)$ form a triangle. The sum of the lengths of two sides of a triangle must be larger than the length of the third side, so $2a > 2c$. Some algebraic manipulation of this equation leads to the *standard form* for an ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1}$$

where $b = \sqrt{a^2 - c^2}$. The argument of the square root is positive since earlier we argued that $a > c$. Moreover, $b < a$ is guaranteed since $b = \sqrt{a^2 - c^2} < \sqrt{a^2} = a$.

The *center* of the standard form ellipse is $(0, 0)$. The *vertices* are $(\pm a, 0)$. The *major axis* is the line segment that connects the vertices. The *minor axis* is the line segment with end points $(0, \pm b)$. The number a is called the *semimajor axis* and the number b is called the *semiminor axis*. [Note: I disagree with the use of the term "axis" to denote length.] The *eccentricity* is the ratio $c/a \in [0, 1]$ and is a measure of how stretched the ellipse is from a circle. A ratio of 0 occurs for a circle. A ratio nearly 1 incidates a long and narrow ellipse.

If the foci are chosen to be $(0, \pm c)$ and the sum of distances is $2b$, the standard form is also given by Equation (1), but now $b > c$ and $a = \sqrt{b^2 - c^2} < b$. The center is still $(0, 0)$, but the vertices are now $(0, \pm b)$, the major axis is the line segment connecting the vertices, the minor axis is the line segment with end points $(\pm a, 0)$, the semimajor axis is b, the semiminor axis is a, and the eccentricity is now defined as the ratio c/b.

If $a = b$, the foci are coincident with the origin $(0, 0)$ and the ellipse is really a circle. The concepts of major and minor axes do not apply here, but the eccentricity is 0.

Area. The *area* of an ellipse in standard form is

$$A = \pi a b. \tag{2}$$

Length. The *length* of an ellipse is the total arc length of the curve. A closed form algebraic solution does not exist, but the length is given by an integral

$$L = 2\int_{-a}^{a} \sqrt{1 + \frac{b^2 x^2}{a^2(a^2 - x^2)}}\, dx = 2\int_{-1}^{1} \sqrt{\frac{1 - (\lambda^2 - 1)t^2}{1 - t^2}}\, dt \tag{3}$$

where $\lambda = b/a$. The integral can be approximated with a numerical integrator.

Center-Orient Form. An ellipse in the standard form given by Equation (1) can be oriented via a rotation so that the major and minor axes are not necessarily parallel to the coordinate axes. In vector/matrix form, the standard form is

$$1 = \frac{x^2}{a^2} + \frac{y^2}{b^2} = \left[\begin{array}{cc} x & y \end{array}\right] \left[\begin{array}{cc} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{array}\right] \left[\begin{array}{c} x \\ y \end{array}\right] =: \mathbf{X}^\mathsf{T} D \mathbf{X} \tag{4}$$

where the last equality defines the 2×1 vector $\mathbf{X} = [x\, y]^\mathsf{T}$, the 2×2 diagonal matrix $D = \mathrm{Diag}(1/a^2, 1/b^2)$, and superscript T denotes the transpose operation.

The ellipse may be rotated to a different orientation by a 2×2 rotation matrix

$$R = \left[\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array}\right]$$

The major axis direction $(1, 0)$ is rotated to $(\cos\theta, \sin\theta)$ and the minor axis direction $(0, 1)$ is rotated to $(-\sin\theta, \cos\theta)$. The general transformation is $\mathbf{Y} = R\mathbf{X}$ with inverse $\mathbf{X} = R^\mathsf{T}\mathbf{Y}$. Substituting this into Equation (4) leads to

$$\mathbf{Y}^\mathsf{T} R D R^\mathsf{T} \mathbf{Y} = 1. \tag{5}$$

After orientation the ellipse can be additionally translated so that its old center, the origin $\mathbf{0}$, is mapped to a new center $\mathbf{K}$. The general transformation is $\mathbf{Y} = R\mathbf{X} + \mathbf{K}$; the rotation R is applied first, followed by the translation $\mathbf{K}$. Equation (5) is modified to include the translation,

$$(\mathbf{Y} - \mathbf{K})^\mathsf{T} R D R^\mathsf{T} (\mathbf{Y} - \mathbf{K}) = 1. \tag{6}$$

General Quadratic Form. When the Equation (6) is expanded and all terms are grouped on the left-hand side of the equation, the resulting polynomial has x, y, x^2, xy, and y^2 terms. The general quadratic equation for an ellipse is

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + b_1 x + b_2 y + c = 0 \tag{7}$$

or in vector/matrix form,

$$\mathbf{Y}^\mathsf{T} A \mathbf{Y} + \mathbf{B}^\mathsf{T} \mathbf{Y} + c = 0 \tag{8}$$

where

$$\mathbf{Y} = \left[\begin{array}{c} x \\ y \end{array}\right], \quad A = \left[\begin{array}{cc} a_{11} & a_{12} \\ a_{12} & a_{22} \end{array}\right], \quad \text{and} \quad \mathbf{B} = \left[\begin{array}{c} b_1 \\ b_2 \end{array}\right].$$

All conic sections are represented by these equations. The ellipses are those for which $a_{11}a_{22} - a_{12}^2 > 0$. Observe that this condition states the determinant of A is positive, so A is an invertible matrix with inverse denoted by A^{-1}. The matrix A and its inverse A^{-1} are both symmetric matrices since $A^\mathsf{T} = A$ and $A^{-T} = (A^\mathsf{T})^{-1} = A^{-1}$.

A typical problem is to start with the general quadratic form and convert to the center-orient form. This can be done by first completing the square on the equation. Consider that

$$\begin{array}{rcl} (\mathbf{Y} - \mathbf{K})^\mathsf{T} A (\mathbf{Y} - \mathbf{K}) & = & \mathbf{Y}^\mathsf{T} A \mathbf{Y} - 2\mathbf{K}^\mathsf{T} A \mathbf{Y} + \mathbf{K}^\mathsf{T} A \mathbf{K} \\ & = & (\mathbf{Y}^\mathsf{T} A \mathbf{Y} + \mathbf{B}^\mathsf{T} \mathbf{Y} + c) - (2A\mathbf{K} + \mathbf{B})^\mathsf{T} \mathbf{Y} + (\mathbf{K}^\mathsf{T} A \mathbf{K} - c) \\ & = & -(2A\mathbf{K} + \mathbf{B})^\mathsf{T} \mathbf{Y} + (\mathbf{K}^\mathsf{T} A \mathbf{K} - c). \end{array}$$

If you set $\mathbf{K} = -A^{-1}\mathbf{B}/2$, then $\mathbf{K}^\mathsf{T} A\mathbf{K} = \mathbf{B}^\mathsf{T} A^{-1}\mathbf{B}/4$ and

$$(\mathbf{Y} - \mathbf{K})^\mathsf{T} A(\mathbf{Y} - \mathbf{K}) = \mathbf{B}^\mathsf{T} A^{-1}\mathbf{B}/4 - c.$$

Dividing by the scalar on the right-hand side of the last equation and setting $M = A/(\mathbf{B}^\mathsf{T} A^{-1}\mathbf{B}/4 - c)$ produces

$$(\mathbf{Y} - \mathbf{K})^\mathsf{T} M(\mathbf{Y} - \mathbf{K}) = 1.$$

Finally, M can be factored using an eigendecomposition into $M = RDR^\mathsf{T}$ where R is a rotation matrix and D is a diagonal matrix whose diagonal entries are positive. The final equation obtained by substituting the factorization for M is exactly Equation (6).

For a 2×2 matrix, the eigendecomposition can be done symbolically. An eigenvector $\mathbf{V}$ of M corresponding to an eigenvalue λ is a nonzero vector such that $M\mathbf{V} = \lambda\mathbf{V}$. The eigenvalues are solutions to the quadratic equation $\det(M - \lambda I) = 0$ where I is the identity matrix. Since M is a symmetric matrix, the eigenvalues must be real numbers. For each eigenvalue, a corresponding eigenvector $\mathbf{V}$ is a nonzero solution to $(M - \lambda I)\mathbf{V} = \mathbf{0}$. Let $M = [m_{ij}]$. The quadratic equation is

$$\begin{aligned} 0 &= \det(M - \lambda I) \\ &= \det \begin{bmatrix} m_{11} - \lambda & m_{12} \\ m_{12} & m_{22} - \lambda \end{bmatrix} \\ &= (m_{11} - \lambda)(m_{22} - \lambda) - m_{12}^2 \\ &= \lambda^2 - (m_{11} + m_{22})\lambda + (m_{11}m_{22} - m_{12}^2). \end{aligned}$$

The roots are

$$\lambda = \frac{(m_{11} + m_{22}) \pm \sqrt{(m_{11} + m_{22})^2 - 4(m_{11}m_{22} - m_{12}^2)}}{2} = \frac{(m_{11} + m_{22}) \pm \sqrt{(m_{11} - m_{22})^2 + 4m_{12}^2}}{2}. \quad (9)$$

The argument of the square root is nonnegative, so the roots must be real-valued. The only way for the roots to be equal is if $m_{11} = m_{22}$ and $m_{12} = 0$, in which case M must have been a scalar multiple of the identity matrix (the ellipse is really a circle). I assume for the remainder of the construction that the two eigenvalues are different.

Define λ_1 to be the eigenvalue in Equation (9) that uses the plus sign and define λ_2 to be the one that uses the minus sign. It is the case that $\lambda_1 > \lambda_2$. An eigenvector corresponding to λ_1 is perpendicular to one of the rows of the matrix

$$\begin{bmatrix} m_{11} - \lambda_1 & m_{12} \\ m_{12} & m_{22} - \lambda_1 \end{bmatrix} = \begin{bmatrix} \frac{(m_{11} - m_{22}) - P}{2} & m_{12} \\ m_{12} & \frac{-(m_{11} - m_{22}) - P}{2} \end{bmatrix}$$

where $P = \sqrt{(m_{11} - m_{22})^2 + 4m_{12}^2} > 0$. We need to be certain that the selected row is not the zero vector. If $m_{12} \neq 0$, then either row will suffice. In a floating-point system, though, m_{12} might be nearly zero. It is better to devise a selection scheme that does not suffer from numerical round-off errors. Specifically, if $m_{11} \geq m_{22}$, then

$$|-(m_{11} - m_{22}) - P| \geq |(m_{11} - m_{22}) - P|$$

The best choice is to use the second row to generate the eigenvector. If $m_{11} \leq m_{22}$, then

$$|-(m_{11} - m_{22}) - P| \leq |(m_{11} - m_{22}) - P|$$

and the best choice is to use the first row to generate the eigenvector. Let $\mathbf{U}_1 = (\alpha, \beta)$ be a normalized vector that is perpendicular to the selected row. The eigenvector corresponding to λ_2 is chosen to be $\mathbf{U}_2 = (-\beta, \alpha)$.

By definition of eigenvectors, $M\mathbf{U}_1 = \lambda_1 \mathbf{U}_1$ and $M\mathbf{U}_2 = \lambda_2 \mathbf{U}_2$. We can write the two equations jointly by using a matrix $R = [\mathbf{U}_1 \ \mathbf{U}_2]$ whose columns are the unit-length eigenvectors. The columns are unit length and perpendicular to each other, so R is an orthogonal matrix. In fact, by the choice of $\mathbf{U}_2$, R happens to be a rotation matrix (no reflection component so to speak). The joint equation is $MR = RD$ where $D = \mathrm{Diag}(\lambda_1, \lambda_2)$. Multiplying on the right by R^T leads to the decomposition $M = RDR^\mathsf{T}$.

In summary, for an ellipse specified as $a_{11}x^2+2a_{12}xy+a_{22}y^2+b_1x+b_2y+c = 0$, first verify that $a_{11}a_{22}-a_{12}^2 > 0$ so that you really do have an ellipse. Then

1. The center is
$$\mathbf{K} = (k_1, k_2) = \frac{(a_{22}b_1 - a_{12}b_2, a_{11}b_2 - a_{12}b_1)}{2(a_{12}^2 - a_{11}a_{22})}. \tag{10}$$

2. Set $\mu = 1/(\mathbf{K}^\mathsf{T} A \mathbf{K} - c) = 1/(a_{11}k_1^2 + 2a_{12}k_1k_2 + a_{22}k_2^2 - c)$ and define $m_{11} = \mu a_{11}$, $m_{12} = \mu a_{12}$, and $m_{22} = \mu a_{22}$.

3. Set $\lambda_1 = ((m_{11} + m_{22}) + \sqrt{(m_{11} - m_{22})^2 + 4m_{12}^2})/2$. The semiminor axis of the ellipse is
$$b = \frac{1}{\sqrt{\lambda_1}}. \tag{11}$$
Set $\lambda_2 = ((m_{11} + m_{22}) - \sqrt{(m_{11} - m_{22})^2 + 4m_{12}^2})/2$. The semimajor axis of the ellipse is
$$a = \frac{1}{\sqrt{\lambda_2}}. \tag{12}$$

4. If $m_{11} \geq m_{22}$, choose the minor axis direction of the ellipse to be
$$\mathbf{U}_1 = \frac{(\lambda_1 - m_{22}, m_{12})}{|(\lambda_1 - m_{22}, m_{12})|} \tag{13}$$
If $m_{11} < m_{22}$, choose the minor axis direction to be
$$\mathbf{U}_1 = \frac{(m_{12}, \lambda_1 - m_{11})}{|(m_{12}, \lambda_1 - m_{11})|} \tag{14}$$
If $\mathbf{U}_1 = (\alpha, \beta)$, choose the major axis direction to be $\mathbf{U}_2 = (-\beta, \alpha)$.

5. If all you need is the angle formed by the major axis with the positive x-axis, that angle satisfies the equation
$$\tan(2\theta) = -\frac{2a_{12}}{a_{22} - a_{11}}$$
This is obtained by making the change of variables $x = \bar{x}\cos\theta - \bar{y}\sin\theta$ and $y = \bar{x}\sin\theta + \bar{y}\cos\theta$ and substituting into the original quadratic equation. After expanding all terms, the coefficient of $\bar{x}\bar{y}$ is
$$-2a_{11}\sin\theta\cos\theta + 2a_{12}(\cos^2\theta - \sin^2\theta) + 2a_{22}\sin\theta\cos\theta = 2a_{12}\cos(2\theta) + (a_{22} - a_{11})\sin(2\theta)$$
Setting this coefficient to zero gives you an axis-aligned ellipse in the $(\bar{x}, \bar{y})$ coordinate system, so the angle θ represents how much you must rotate the original ellipse to the axis-aligned one.

6. For $R = [\mathbf{U}_1\ \mathbf{U}_2]$ where $\mathbf{U}_1$ and $\mathbf{U}_2$ are written as columns and $D = \mathrm{Diag}(1/a^2, 1/b^2)$, the ellipse is represented by the factored form

$$(\mathbf{Y}-\mathbf{K})^{\mathsf{T}} R D R^{\mathsf{T}} (\mathbf{Y}-\mathbf{K}) = (\mathbf{Y}-\mathbf{K})^{\mathsf{T}} \left(\frac{1}{a^2}\mathbf{U}_1\mathbf{U}_1^{\mathsf{T}} + \frac{1}{b^2}\mathbf{U}_2\mathbf{U}_2^{\mathsf{T}} \right) (\mathbf{Y}-\mathbf{K}) = 1. \tag{15}$$

7. Observe that $\mathbf{Y} = \mathbf{K} + R\mathbf{X} = \mathbf{K} + x\mathbf{U}_1 + y\mathbf{U}_2$. Replacing this in the factored form leads to $(x/a)^2 + (y/b)^2 = 1$, as expected since originally $\mathbf{Y}$ was selected to be the coordinates representing the rotation and translation of the standard form ellipse with coordinates $\mathbf{X}$.

8. The bounding rectangle for the ellipse that has the same directions as the major and minor axes of the ellipse has center $\mathbf{K}$. The four corners are $\mathbf{K} \pm a\mathbf{U}_1 \pm b\mathbf{U}_2$.